On Time versus Size for Monotone Dynamic Monopolies in Regular Topologies [1]

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ABSTRACT: We consider a well known distributed coloring game played on a simple connected graph: initially, each vertex is colored black or white; at each round, each vertex simultaneously recolors itself by the color of the simple (strong) majority of its neighbours. A set of vertices M is said to be a *dynamo*, if starting the game with only the vertices of M colored black, the computation eventually reaches an all-black configuration.

The importance of this game follows from the fact that it models the spread of faults in point-to-point systems with majority-based voting; in particular, dynamos correspond to those sets of initial failures which will lead the entire system to fail. Investigations on dynamos have been extensive but restricted to establishing tight bounds on the size (i.e. how small a dynamic monopoly might be).

In this paper we start to study dynamos systematically with respect to both *the size* and *the time* (i.e. how many rounds are needed to reach all-black configuration) in various models and topologies.

We derive tight tradeoffs between the size and the time for a number of regular graphs, including rings, complete d-ary trees, tori, wrapped butterflies, cube connected cycles and hypercubes. In addition, we determine optimal size bounds of irreversible dynamos for butterflies and shuffle–exchange using simple majority and for DeBruijn using strong majority rules. Fi-

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nally, we make some observations concerning irreversible versus reversible monotone models and slow complete computations from minimal dynamos.

Keywords: Distributed Computing, Majority Rule, Fault Tolerance

1 Introduction

In distributed computing on point-to-point networks, faulty processors can cause the incorrect behaviour of their neighbours. To restrict their influence and to limit the overall damage caused by faulty processors, the idea of majority–based voting schemes is frequently employed. Voting schemes have been used as a decision tool in a number of consensus and agreement protocols, inconsistency resolution protocols in distributed database management systems, data consistency protocols in quorum systems, mutual exclusion algorithms, key distribution in security, reconfiguration under catastrophic faults in system level analysis and computational models in discrete–time dynamical systems. However, in point-to-point systems, majority-based voting can be impotent to stop the propagation of a faulty behaviour started by (sometimes few but) well placed faulty elements. The dynamics, with respect to faults, of a system employing majority-based voting is best described as a synchronous vertex-coloring game on graphs.

Let G be the simple connected graph modelling the topology of the system. Consider the following game played on G using the set of colors $\{black, white\}$. The game proceeds in synchronous rounds. Initially, vertices are colored black (faulty) or white (non-faulty). At each round, each vertex simultaneously considers the current colors of its neighbours, and takes as its new color the one held by the majority of its neighbours. We distinguish between *simple* and *strong* majority, to describe whether or not a white vertex can be colored black in case of equal number of white and black neighbours, respectively. We also distinguish between *irreversible* and *reversible* rules to describe whether the initial faults are permanent or can be mended by the majority rule.

Certain variants of the game were studied in the literature, mainly in the context of discrete–time dynamical systems. The typical problems being studied in this setting involved periodic behaviour of the computation.

In this paper we concentrate on the cases in which the computation converges into the all–black monochromatic configuration; this corresponds to the situation when the entire system will have a faulty behaviour. A set of vertices M is said to be *a dynamic monopoly*, shortly *dynamo*, if starting the game with only the vertices of M colored black, the computation eventually reaches an all-black configuration. Several models of dynamic monopolies have been considered, depending on the majority rule (simple or strong), and on the type of initial faults (reversible or irreversible). Given a dynamo M, let M_t denote the set of black-colored vertices after round t (with $M_0 = M$). A dynamo M is said to be *monotone* if $M_t \subseteq M_{t+1}$ for every $t \geq 0$. It is easy to see that irreversible dynamos are always monotone; in reversible models, the investigation has concentrated only on monotone dynamos.

By *the size* of a dynamo we mean the number of its black vertices; by *the time* of a dynamo we mean the number of rounds needed to reach the all-black configuration.

The size of dynamo is clearly a crucial parameter: a "larger" size implies a less likely occurrence; thus, a system in which the smallest dynamo is large has a high degree of fault-tolerance.

Not surprising, the existing literature has concentrated on the minimum size possible for monopolies. The combinatorial question of bounding the size of monopolies was firstly studied in the static case, where *a static monopoly*, shortly *stamo*, is defined as a set that reaches the final monochromatic configuration in at most one round. Bounding the size of such monopolies, in a variety of models, was the topic of [2, 8, 9, 10]. The investigation has been then extended to the analysis of the size of monotone dynamic monopolies; in this case, all the research has concentrated on monotone dynamos. In this context, there is the study of monopolies which reach the all black configuration in at most two steps [8]. All the other investigations on dynamic monopolies (see references [3, 4, 5, 7, 10]) has focused on determining tight bounds on the size of dynamos, irrespective of their time.

However, the size is not the only aspect of the quality of dynamos. In particular, the time needed for a dynamo to converge into the monochromatic configuration is a very important characteristic, not only from a combinatorial but also from a practical point of view. For example, if a catastrophic set of faults (i.e., a dynamo) has a "slow" evolution, its presence might be more easily detected (and external action be taken); on the other hand, a "fast" dynamo is inherently more dangerous for the system.

These two measures, size and time, are clearly related; the nature of their relationship is one of trade-off.

In this paper we start the analysis of the time–size tradeoff for monotone dynamos. In particular, we study the combinatorial problem of determining the size of dynamic monopolies of t–time bounded computations for given t. In this light, the previous investigations provided results on the size of 1-, 2- and ∞-time dynamos.

We try to give answers to the following questions (for specific topologies and various models of dynamic monopolies):

- what is the minimal size of a stamo (1-time dynamo) ?
- what is the minimal time among the minimal-size ∞-time dynamos ?
- what is a general time–size tradeoff for dynamos ?

This paper contains not only the first systematic analysis of the relation between size and time of dynamos, but also a large number of specific results are presented on a variety of networks. We derive tight tradeoffs between the size and the time for a number of specific regular topologies, including rings, complete d-ary trees, tori, wrapped butterflies, cube connected cycles and hypercubes. In addition, we determine asymptotically optimal size bounds of irreversible dynamos for butterflies and shuffle–exchange using simple majority and for DeBruijn using strong majority rules. Finally, we make some observations concerning irreversible versus reversible monotone models and slow complete computations from minimal dynamos.

2 Definitions

Basic definitions are from [3, 4, 5]. We use also the following notions.

Let $G = (V, E)$ be a simple connected graph and $N = |V|$. Assume the vertices of the graph G be colored by black or white. By a *configuration* on G we mean a partition of V into the set of black and the set of white vertices. (For simplicity, a configuration will be referred as the set of its black vertices.) By the *size* of a configuration we mean the number of its black vertices.

The *final (all-black) configuration* of G is the configuration in which all vertices of G are black. A *catastrophic configuration* is a configuration from which there is a synchronous computation (every round of which is performed by means of simple (or strong) majority rule applied to each vertex of G simultaneously) leading to the final configuration. Such a computation is called a *complete computation*. By *time complexity* of a complete computation we mean the number of its rounds. By t-time computation we mean a complete computation with t rounds. A *dynamo* is a set of black nodes whose corresponding configuration is catastrophic. By t–time dynamo we mean a dynamo having complete computation with t rounds. By a simple (strong) dynamo we mean a dynamo using simple (strong) majority rule. If not mentioned explicitely, by a dynamo we mean a dynamo using simple majority rule. By a minimal dynamo we mean a dynamo of minimal size. Given a dynamo M, let M_t denote the set of black-colored vertices after round t (with $M_0 = M$). A dynamo M is *monotone* if $M_t \subseteq M_{t+1}$ for every $t \geq 0$. It is easy to see that irreversible dynamos are always monotone. In reversible models, we concentrate only on monotone dynamos.

An *immune subgraph* of G is a subgraph for which there does not exist a complete computation on G, starting from some initial configuration having all vertices of immune subgraph white. By t-immune subgraph we mean an immune subgraph such that there does not exist a t-time computation on G, starting from some initial configuration having all vertices of t-immune subgraph white.

3 Results on Time versus Size

3.1 Lower bounds on the size of dynamos in bounded degree graphs

In this subsection we present certain useful lower bound results on the size of dynamos, in terms of vertex set cardinality N, vertex degree d and time bound t. Some of these results are applied to special topologies in the following subsections.

We use the following proposition, due to [11], applicable to arbitrary graphs:

THEOREM 3.1
In each configuration C of a complete computation on G using strong majority rule, the number of white vertices in C is upper bounded by the number of edges joining vertices of different colors in C.

PROOF. Consider a complete computation $C = C_0, C_1, ..., C_t$. As a basis of backwards induction note that the configuration C_t consists of all black vertices. That means that initially the proposition of the theorem holds. Suppose the statement holds for C_i. Let M_i be the set of vertices colored black in the i-th step. The white vertices in C_{i-1} are those from C_i plus the vertices from M_i. In C_{i-1}, each vertex from M_i has strict majority of black neighbours, hence the number of edges joining vertices of different colors has decreased from C_{i-1} to C_i by at least $|M_i|$. ■

THEOREM 3.2
Let $G = (V, E)$ be a graph of maximum degree d, where $|V| = N$. The size of a minimal strong irreversible dynamo of G is at least $\lceil \frac{N}{d+1} \rceil$.

PROOF. Let M be a minimal strong irreversible configuration of a complete computation on G. As the previous theorem holds also for initial configuration M we have

$$|V - M| \leq d \cdot |M|$$

and so

$$\frac{|V|}{d+1} \leq |M|$$

■

COROLLARY 3.3
Let $G = (V, E)$ be an odd degree graph of maximum degree d, where $|V| = N$. The size of a minimal irreversible dynamo of G is at least $\lceil \frac{N}{d+1} \rceil$.

PROOF. The notions of strong and simple majority are identical, as all vertices of G have odd degree. ■

We now turn our attention to time bounded complete computations. We give a lower bound on the size of irreversible dynamos on d-regular graphs having complete computations bounded by time t (t–time dynamos).

THEOREM 3.4
Let $G = (V, E)$ be a regular graph, where $|V| = N$. The size of a minimal t–time irreversible dynamo of G is at least $\lceil \frac{N}{2t+1} \rceil$.

PROOF. Let G be a regular graph of degree d. Let M_0 be a minimal irreversible dynamo for G, say of size m (i.e. $|M_0| = m$). Take an arbitrary complete computation $M_0 = C_0 \mapsto C_1 \mapsto ... \mapsto C_t$, where C_i is the set of all black vertices in G at round i.

First, consider $d = 2\delta$. The number of edges outgoing from the dynamo C_0 does not increase at round 1, i.e. the number of edges outgoing from C_1 is not greater than that leaving C_0. We now estimate the maximal number of vertices that can be colored black at round 1. For each vertex v colored black at the 1-st round, the number of its edges leading into C_0 is at least δ. So there are at most $2m$ vertices colored black at the first round of the above complete computation. This argument holds for each round of the complete computation. At the t-th round of the complete computation, the whole graph G is colored black. Hence, it holds $m + 2mt \geq N$, and so $m \geq \lceil \frac{N}{2t+1} \rceil$.

The case $d = 2\delta + 1$ is handled similarly. ■

The next lower bound is suitable for near-regular graphs.

COROLLARY 3.5
Let Δ be the ratio of the maximum to the minimum vertex degree in G. Then the size of a minimal t–time simple irreversible dynamo of G is at least $\lceil \frac{N}{2\Delta t+1} \rceil$.

For regular odd–degree graphs, a slightly better lower bound for irreversible stamo can be obtained.

THEOREM 3.6
Let $G = (V, E)$ be a regular graph of degree $2d + 1$, where $|V| = N$. Then the size of a minimal irreversible static monopoly is at least $\left\lfloor \frac{d+1}{3d+2} \cdot N \right\rfloor$.

PROOF. Let m denote the size (i.e. the number of black vertices) of a 1-time irreversible dynamo. For each white vertex, there must be at least $d + 1$ adjacent black vertices. One black vertex can be adjacent to at most $2d + 1$ white vertices. Thus we have $(N - m)(d + 1)/(2d + 1) \leq m$ and the result follows. ∎

We now consider monotone reversible dynamos on planar graphs. In [6] it was proven that every 1-time reversible dynamo in an N-vertex planar graph has the size $\Omega(N)$. We extend this result as follows:

THEOREM 3.7
Every t-time simple reversible monotone dynamo in a N-vertex planar graph has the size at least $\frac{N-2}{6t+1} + 2$.

PROOF. Consider a fixed m-vertex dynamo M. We shall bound the number of vertices of a graph G, $G \supseteq M$, on which a t–time complete computation of M exists. Let E_i denote the edges with endpoints of different colors after $i - 1$ rounds. Clearly $|E_i| \leq |E_{i-1}|$ for each i, $1 < i \leq t$. Let V_i denote the vertices colored in the i-th round. Because $|V_i| \leq |E_i|$ we get $N = m + \sum_{i=1}^{t} |V_i| \leq m + t|E_1|$. Let $deg_M(v)$ be the degree of the vertex v in the subgraph of G induced by M. It holds $|E_1| \leq \sum_{v \in M} deg_M(v)$ and because G (and hence M) is planar, it holds $\sum_{v \in M} deg_M(v) \leq 6m - 12$ which gives the result. ∎

For rings R_N a trivial upper bound on the size of 1–time reversible dynamo is $\frac{N}{1.5}$. However, this bound can be improved down to $\frac{N+12}{7}$ for special constructed planar graphs. This means that for 1–time simple reversible monotone dynamos the lower bound given in the previous theorem is close to the above upper bound. The construction is as follows. Take an arbitrary triangulation with $|V|$ vertices and $|E|$ edges. To each vertex v add $deg(v)$ additional neighbours of degree 1. The basis graph creates a 1-time dynamo. The number of added vertices is $\sum deg(v) = 2|E| = 6|V| - 12$, so altogether there are $N = 7|V| - 12$ vertices and the dynamo has size $|V|$.

Note that for the irreversible model, for every $D \leq N$ there are planar graphs with diameter D and N vertices having an irreversible dynamo of size $\lceil D/(2t + 1) \rceil$. Consider a path of length $D - 1$ and add $N - D + 1$ additional vertices to one of the endpoints of the path. The obtained graph has N vertices, diameter D and the dynamo is of size $\lceil D/(2t + 1) \rceil$.

THEOREM 3.8
Every t–time reversible monotone dynamo in an N-vertex d-regular planar graph has the size at least $\frac{N}{1+td/2}$.

PROOF. From the proof of Theorem 3.7 it follows that $N \leq m + t|E_1|$. Since $deg_M(v) \geq d/2$, we have that $|E_1| \leq md/2$. ■

3.2 Complete trees

By $T_{d,h}$ denote the complete d-ary tree of height h. Let $N = \frac{d^{h+1}-1}{d-1}$ be the number of vertices. There is a unique characterization of minimal irreversible dynamos in $T_{d,h}$ provided by the following "folk" theorem:

THEOREM 3.9
The configuration C in $T_{d,h}$ forms a minimal irreversible dynamo if and only if C consists of exactly all parents of leaves in $T_{d,h}$.

PROOF. Let C be a set of all vertices, being parents of leaves in $T_{d,h}$. Clearly, C forms an irreversible dynamo. We need to show that C is a minimal dynamo and that it is unique. To show the first property, take d leaves with corresponding parent as an induced $(d+1)$-vertex subgraph. Each of these subgraphs is immune. Thus any dynamo requires at least one initially black vertex in it. The uniqueness follows from the fact that each of these immune subgraphs requires 1 black vertex only if it is the parent of the leaves. ■

COROLLARY 3.10
The size of a minimal simple irreversible dynamo in the complete d-ary tree $T_{d,h}$ is $\frac{N(d-1)+1}{d^2}$. The time to reach the all-black configuration from a minimal irreversible dynamo is $h-1$.

PROOF. The size of a minimal dynamo follows from Theorem 3.9. Clearly, the dynamo converges in time $h-1$. On the other hand, from the same "immune subgraph argument" used in the proof of Theorem 3.9, the time $h-1$ is also necessary to reach the final configuration from a minimal dynamo in $T_{d,h}$. ■

THEOREM 3.11
The size of a minimal t–time simple irreversible dynamo in complete tree $T_{d,h}$ is $\frac{d^{h+t}}{d^{t+1}-1}\left(1 - \frac{1}{d^{(t+1)\lceil\frac{h}{t+1}\rceil}}\right)$.

PROOF.

Let us call level l ($0 \leq l \leq h$) the set of vertices in $T_{d,h}$ with distance l from the root.

Let us call strip i ($0 < i < \frac{h}{t+1}$) the union of $t+1$ consecutive levels starting from level $h-(t+1)i$. Strip $\left\lceil \frac{h}{t+1} \right\rceil$ is the union of $h-(t+1)\left(\left\lceil \frac{h}{t+1} \right\rceil - 1\right)$ levels starting from level 0 and is called the last strip. The last strip is a complete d-ary tree and we denote its height t'.

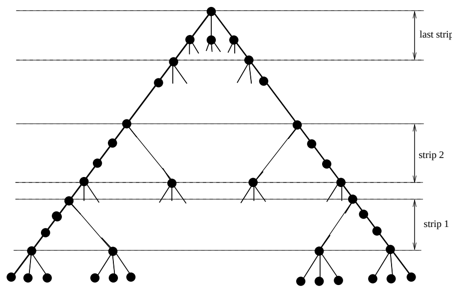

Fig. 1

Strips are pairwise disjoint and every strip consists of a set of disjoint complete d-ary trees.

Upper bound: Consider a configuration where in each strip the black vertices are leaves of all trees in that strip. In all strips but the last strip the ratio of initially black vertices is $\frac{d^t(d-1)}{d^{t+1}-1}$. The last strip contains $\frac{d^{t'+1}-1}{d-1}$ vertices, $d^{t'}$ of which are black. So the number of black vertices is $\left(N - d^h - \frac{d^{t'+1}-1}{d-1}\right)\frac{d^t(d-1)}{d^{t+1}-1} + d^{t'} = \frac{d^{h+t}}{d^{t+1}-1}\left(1 - \frac{1}{d^{(t+1)\left\lceil \frac{h}{t+1} \right\rceil}}\right)$.

Lower bound: Let us call the dynamo from above M. We prove that M is minimal. Consider a minimal dynamo $M^{(0)}$. We shall transform $M^{(0)}$ to M without adding black vertices. We use induction on the number of strips. As $M^{(0)}$ is a dynamo every $d+1$ vertex subgraph consisting of a leaf of a tree in strip 1 together with its d sons contains at least one black vertex. We can move all black vertices in these subgraphs to the leaves of trees in strip 1. Without affecting the time of the dynamo, we can move all black non-leaf vertices in strip 1 upwards to the leaves of trees in strip 2. Thus we have a dynamo $M^{(1)}$ which is from strip 1 below identical with M.

Now suppose we have a dynamo $M^{(i)}$ which is from strip i below identical with M. Consider trees of height t rooted in the last level of strip $i+1$. These are t-immune and disjoint, so every one of them contains at least one vertex of $M^{(i)}$. Because strip i in $M^{(i)}$ is identical with strip i in M, it follows that in $M^{(i)}$ all vertices on the last level of strip $i+1$ are black. Hence, without affecting the computation time we can move all non-leaf vertices from strip $i+1$ upwards. The result follows. ∎

COROLLARY 3.12

The size of a minimal simple irreversible stamo in complete d-ary tree $T_{d,h}$ is $\left\lceil \frac{N-1}{d+1} \right\rceil$.

PROOF. From the previous theorem, for h even we get the size of 1-time dynamo to be $\frac{N-1}{d+1}$. For h odd we get $\frac{N-1}{d+1} + \frac{d-1}{d+1}$. As the size is integral, the result follows. ∎

3.3 Rings

For N–vertex rings R_N we also know exact characterization of minimal irreversible dynamos. We now give exact time–size tradeoff for irreversible model with simple majority rule.

THEOREM 3.13
The size of a minimal t–time simple irreversible dynamo in R_N is $\lceil \frac{N}{2t+1} \rceil$. The time to reach all-black configuration from a minimal irreversible dynamo is $\lceil \frac{N-1}{2} \rceil$.

PROOF. The lower bound on the size follows from Theorem 3.4. The lower bound on the time follows from Theorem 3.4 and the fact that the size of minimal irreversible dynamo is 1. ∎

3.4 Tori

An $n \times m$ tori $T_{n,m}$ consists of $n \cdot m$ vertices, where each vertex $v_{i,j}$ with $0 \leq i \leq n-1, 0 \leq j \leq m-1$ is connected to the four vertices $v_{(i-1) mod\ n,j}$, $v_{(i+1) mod\ n,j}$, $v_{i,(j-1) mod\ m}$, $v_{i,(j+1) mod\ m}$.

The first part of the following result is from [4].

THEOREM 3.14
The size of a minimal simple irreversible dynamo in 2-dimensional tori $T_{n,m}$ is $\lceil n/2 \rceil + \lceil m/2 \rceil - 1$. The time of a complete computation from a minimal dynamo in tori $T_{n,m}$ is at least $\frac{1}{2}\left(\frac{mn}{\lceil m/2 \rceil + \lceil n/2 \rceil - 1} - 1\right)$.

The second part of the previous result can be obtained by combining the size formula in the previous theorem with Theorem 3.4. We get $\lceil m/2 \rceil + \lceil n/2 \rceil - 1 \geq \left\lceil \frac{N}{2t+1} \right\rceil$. Further, it can be shown that the previous time bound is tight. Consider the "diagonal" dynamo. Its behaviour matches the estimates from the proof of Theorem 3.4.

THEOREM 3.15
The size of a minimal t-time simple irreversible dynamo in 2-dimensional tori $T_{n,n}$ is $\left\lceil \frac{N}{2t+1} \right\rceil$.

PROOF. For the upper bound consider a dynamo consisting of diagonals in $T_{n,n}$ of distance $2t+1$. The size of this dynamo is $\left\lceil \frac{N}{2t+1} \right\rceil$ and the time is t. Following Theorem 3.4 the lower bound is $\lceil \frac{N}{2t+1} \rceil$. ∎

COROLLARY 3.16
The size of a minimal simple irreversible stamo in a 2-dimensional tori is $\lceil N/3 \rceil$.

3.5 Butterfly

The butterfly graph of dimension n (denoted as $BF(n)$) consists of $n+1$ columns, each column containing 2^n vertices, each of them labeled with unique binary string of the length n. An edge connects two vertices in $BF(n)$ if and only if they are in the consecutive i-th and $(i+1)$-st columns, respectively, and their labels are either equal or differ only in the i-th bit.

Wrapped butterfly graph of dimension n (denoted as $WBF(n)$) is obtained from $BF(n)$ by making the first and the last column identical. $WBF(n)$ is a regular graph of degree 4.

The following results on sizes of simple irreversible dynamos in BF and WBF are from [7].

THEOREM 3.17
The size of a minimal simple irreversible dynamo in $BF(n)$ is at least $2^{\lfloor \frac{n-1}{2} \rfloor}$ and at most 2^{n-2}. The size of a minimal simple irreversible dynamo in $WBF(n)$ is at least $2^{\lfloor \frac{n}{2} \rfloor}$ and at most $2^{n-2}+2^{n-3}+2^{n-4}$.

Now we provide an improvement of the lower bound for BF in Theorem 3.17.

$BF(n)$ can be sketched such that 2^n rows are numbered top–down in order $0, ..., 2^n - 1$ and $n+1$ columns are numbered right–left in order $0, ..., n$.

LEMMA 3.18
Consider an arbitrary dynamo M in $BF(n)$. Then M can be transformed into a dynamo M' of at most the same size such that black vertices are only in rows $0, ..., 2^{n-1} - 1$ and in columns $0, ..., n-1$.

PROOF. Black vertices of M in the row i and the column n are moved to the row i and the column $n-1$. Black vertices of M in the row $2^{n-1}+k$ are moved to the row k (in the same column j, $j<n$). It is easy to verify that a new configuration is complete. ■

Let $BF'(n)$ be a multigraph obtained from $BF(n)$ by adding a loop to every vertex in column n. We generalize the notion of a dynamo in a natural way to multigraphs (i.e. in $BF'(n)$ a white vertex in column n has to have two black neighbours in order to become black).

Consider a dynamo M in $BF'(n-1)$. Clearly each irreversible computation is monotone. Therefore to each complete computation C from a dynamo M can be assigned an acyclic dependency digraph $G_{M,C}$ as follows. If a vertex v in $BF'(n-1)$ has been colored in C on the basis of a black-colored neighbour w, we add a directed edge (w,v).

LEMMA 3.19
To each vertex u in the column 0 of $BF'(n-1)$ can be assigned a path $u=u_0, u_1, u_2$, ..., u_k such that

1. vertices $u_0, ..., u_{k-1}$ are white, u_k is black (in the dynamo M);
2. the column of u_{i+1} is greater than the column of u_i (i.e. the path is "one-directional")
3. paths $u_0, u_1, ..., u_k$ and $v_0, v_1, ..., v_m$ starting from different vertices u, v in column 0 are edge disjoint.

PROOF. Consider a complete computation from a dynamo M in $BF'(n-1)$. Denote its dependency digraph as G.

The path for a vertex u in the column 0 is constructed as follows. If u is black (i.e. $u \in M$), there is a path of length 0 in $BF'(n-1)$. Otherwise, u was colored on the basis of some of its neighbours, say u_1. If $u_1 \in M$, there is a path of length 1 in $BF'(n-1)$ (and this path is edge disjoint with an arbitrary one-directional path in $BF'(n-1)$ starting from a vertex in column 0). If $u_1 \notin M$, it must have been colored on the basis of two neighbours, so there are two arcs in G ingoing to u_1. As G is acyclic and the arc (u_1, u) is in G, at least one of the ingoing arcs leads to the left and we choose this one. Iterating this procedure a path in $BF'(n-1)$ can be constructed ending with a vertex from M (if we approach the leftmost column in $BF'(n-1)$, we can use the fact that a vertex in column $n-1$ needs two neighbours black to be recolored).

Now we must show that the chosen paths are edge disjoint. Assume we have constructed a prefix $u_0, u_1, ..., u_i$ of some path such that it is edge disjoint to all already constructed paths. If $u_i \in M$ the path ends in u_i. So let $u_i \notin M$. If u_i does not belong to a previously constructed path, clearly the edge (u_i, u_{i+1}) does not belong to a previously constructed path either. If $u_i = v_j$ for some path $v_0, v_1, ..., v_j, ...$, then this path is unique (clearly there are at most two vertex disjoint paths going via u_i and the existence of two such paths contradicts the fact that u_{i-1}, u_i is in already constructed disjoint section). To continue consider the two arcs going to the vertex u_i in G. One of them is going from v_{j+1}. Two outgoing arcs are going to the right to vertices u_{i-1} and v_{j-1} (again, these vertices are different from the induction hypothesis). So the other neighbour, say w, contributing to recoloring u_i is positioned left and the edge (w, u_i) does not belong to previously constructed paths, so it can be added to the path $u_0, u_1, ..., u_i, w$. ∎

THEOREM 3.20
The size of a minimal simple dynamo in $BF(n)$ is at least 2^{n-2}.

PROOF. Following Lemma 3.18 for each dynamo in $BF(n)$ there exists a dynamo in $BF'(n-1)$ with at most the same size. Following Lemma 3.19 in every dynamo M in $BF'(n-1)$ there are 2^{n-1} one-directional edge disjoint paths ending in black vertices (from M). However at most two paths can end in one black vertex (all paths ending in black vertex are approaching from right neighbours and there are just two edges from the right neighbours). ∎

LEMMA 3.21
The size of a minimal simple irreversible stamo in $BF(n)$ is at least $\lfloor \frac{n+1}{4} \rfloor \cdot 2^n$ and at most $\lceil \frac{n+1}{3} \rceil \cdot 2^n$.

PROOF. The lower bound follows from the decomposition of $BF(n)$ into disjoint cycles of length 4. The upper bound follows from grouping $n+1$ columns of $BF(n)$ into $\lceil \frac{n+1}{3} \rceil$ groups of at most 3 consecutive columns. ∎

Question 1: Interesting question is to determine the size of a minimal simple irreversible dynamo in WBF.

In the following theorem we give a minimal t–time simple irreversible dynamo in WBF having $Time \times Size = \Theta(N)$.

THEOREM 3.22
The size of a minimal t–time simple irreversible dynamo in $WBF(n)$ is $\lceil \frac{N}{2t+1} \rceil$.

PROOF. The lower bound follows from Theorem 3.4. The upper bound follows from the fact that, if the column of 2^n vertices is black, then also its right and left column will be colored black in the next round. ∎

3.6 Cube Connected Cycles

The *cube-connected-cycles* graph of dimension n (denoted as $CCC(n)$) is a regular graph of degree 3 with $n2^n$ vertices. It can be obtained from an n-dimensional hypercube by replacing each vertex of the hypercube with a circle of length n (called a *supernode*), and appropriately connecting the corresponding vertices. Each vertex in $CCC(n)$ can be uniquely labeled with a binary string of length n and an index, called *cursor's position*, in this string.

Given a string $\alpha \in \{0,1\}^n$ and an index j ($0 \leq j \leq n-1$), let $\alpha|j$ denote α with the cursor in position j. The operations of shifting the cursor cyclically to the left and to the right on α are denoted as $L(\alpha|j)$ and $R(\alpha|j)$, respectively. Let $S_j(\alpha) = a_0a_1...\hat{a}_j...a_{n-1}$, where $\hat{a}_j = 1 - a_j$ denote the "shuffle" of the j-th position of $\alpha = a_0a_1...a_{n-1}$; for brevity, let $S(\alpha|j)$ denote $S_j(\alpha)|j$. An edge connects two vertices u, v if and only if v can be obtained from u by means of cyclically shifting the cursor to the left or right or by changing the bit pointed by cursor.

The following bounds on the size of an irreversible dynamo in CCC were established in [7].

THEOREM 3.23
The size of a minimal irreversible dynamo in $CCC(n)$ is at least $max\{\lfloor \frac{n+1}{2} \rfloor \cdot 2^{n-2}, 2^n\}$ and at most $n \cdot 2^{n-2} + 2^{n-3}$.

We now introduce a modular construction for monotone irreversible dynamos; this construction will generate a trade-off between time and size.

Informally, the construction is as follows. Assume for simplicity of presentation that $p = 4(r+1)$ divides n, for some $r > 0$.

We first divide the vertices $\alpha|0, \alpha|1, \ldots, \alpha|n-1$ of each supernode α into consecutive groups of size p, starting with $\alpha|0$. Then, the vertices of a group $\alpha|pi, \alpha|pi+1, \ldots, \alpha|pi+p-1$, ($0 \leq i \leq n/p - 1$) are consecutively paired, starting with $\alpha|pi$. Finally, every pair is given a type (A, B, or C) as follows: in each group, the pair $\langle \alpha|pi, \alpha|pi+1 \rangle$ is assigned type A, the pair $\langle \alpha|pi + p/2 - 1, \alpha|pi + p/2 \rangle$ type C, and all others type B; hence, the pairs in each group will have the following type pattern: AB^rCB^r.

The typing is such that, every pair $\langle \alpha|j, \alpha|j+1 \rangle$ has the same type as the three pairs $\langle S_j(\alpha)|j, S_j(\alpha)|j+1 \rangle$, $\langle S_{j+1}(\alpha)|\, j, S_{j+1}(\alpha)|j+1 \rangle$, and $\langle S_{j+1}(S_j(\alpha))|j, S_{j+1}(S_j(\alpha))$ $|j+1 \rangle = \langle S_j(S_{j+1}(\alpha))|j, S_j(S_{j+1}(\alpha))|j+1 \rangle$ in distinct supernodes, where j is always even; moreover, the nodes in these four pairs form a cycle of lenght 8. As a final step, each of these cycles is "colored" according to its type.

Let us now describe the construction formally. Let $P[j] = \{\alpha \in \{0,1\}^n : \alpha_j = \alpha_{j+1} = 0\}$. Given $\alpha \in P[j]$, let $C[\alpha|j] = \widehat{C_0}, \ldots, \widehat{C_7}$ be the cycle of vertices of

CCC(n) defined by:

$$\widehat{C_{2i}} = \begin{cases} \alpha|j & \text{if } i = 0 \\ R(\widehat{C_{2i-1}}) & \text{if } i = 1,3 \\ L(\widehat{C_{2i-1}}) & \text{if } i = 2 \end{cases}$$

$$\widehat{C_{2i+1}} = S(\widehat{C_{2i}})$$

Let $C[j] = \{C[\alpha|j] : \alpha \in P[j]\}$. Clearly, all these cycles are vertex-disjoint; moreover they cover all strings (i.e., $\forall\beta \in \{0,1\}^n, \exists\alpha \in P[j], \beta \in C[\alpha|j]$). Notice that, for any j, the cardinality of $C[j]$ is $\frac{2^n}{4}$.

Let *A-Color*$[j]$ denote the process of coloring each cycle in $C[j]$ alternating black and white. Note that every cycle so colored will become black in one step.

Let *B-Color*$[j]$ denote the process of coloring each cycle $C[\alpha|j] = \widehat{C_0}, \ldots, \widehat{C_7}$ in $C[j]$ as follows: color $\widehat{C_2}$ and $\widehat{C_6}$ black, all other white.

Let *B'-Color*$[j]$ denote the process of coloring each cycle $C[\alpha|j] = \widehat{C_0}, \ldots, \widehat{C_7}$ in $C[j]$ as follows: color $\widehat{C_0}$ and $\widehat{C_4}$ black, all other white.

Let *C-Color*$[j]$ denote the process of coloring in each cycle of $C[j]$ one arbitrary vertex black and all the others white.

Modular Construction: Let $p = 4(r+1)$ divide n.
For $0 \le k \le \frac{n}{p} - 1$

A-Color(kp)

B-Color$(kp + 2l)$ $(1 \le l \le r)$

B'-Color$(kp + 2l)$ $(r + 2 \le l \le 2r + 1)$

C-Color$(kp + 2(r+1))$

We now prove the correctness of the modular construction and evaluate the time of the resulting dynamos.

LEMMA 3.24
Every cycle in $C[kp]$ becomes black at time 1.

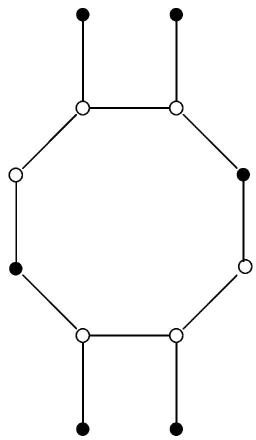

Fig. 2

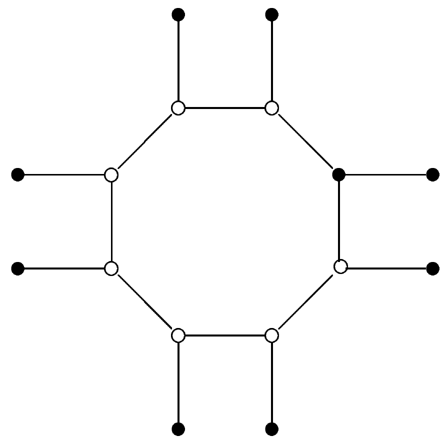

Fig. 3

LEMMA 3.25
Every cycle in $C[kp + 2l]$ $(1 \leq l \leq r)$ will become black in $3l + 1$ time steps.

PROOF. Consider an arbitrary cycle $\widehat{C_0}, \ldots, \widehat{C_7} \in C[kp + 2l]$ $(1 \leq l \leq r)$. By construction $(B - Color)$, $\widehat{C_2}$ and $\widehat{C_6}$ are both black at time 0. Observe that each of $\widehat{C_0}, \widehat{C_1}, \widehat{C_4}$ and $\widehat{C_5}$ is connected to a node in $C[kp + 2(l - 1)]$. Hence, at time $t = 1$, every cycle in $C[kp + 2]$ will be as depicted in Fig. 2; it is easy to verify that, after an additional 3 steps, all nodes in the cycle will become black. Using this as a basis, a simple inductive argument yields the claim. ∎

LEMMA 3.26
Every cycle in $C[kp+2l]$ $(r+1 \leq l \leq 2r+1)$ will become black in $3(2r-l+2)+1$ time steps.

PROOF. Since k is arbitrary, Lemma 3.24 holds also for $C[(k + 1)p]$. By an argument specular to the one of the proof of Lemma 3.25 (using $C[(k + 1)p]$ instead of $C[kp]$ as the black set at time t=1), the claim follows. ∎

LEMMA 3.27
Every cycle in $C[kp + 2(r + 1)]$ will become black in $3r + 5$ time steps.

PROOF. Every cycle $\widehat{C_0}, \ldots, \widehat{C_7} \in C[kp+2(r+1)]$ contains one black vertex by construction $(C - Color)$. Moreover, each of the vertices $\widehat{C_0}, \widehat{C_1}, \widehat{C_4}$ and $\widehat{C_5}$ is connected to a vertex in a cycle in $C[kp + 2r]$; at the same time, each of the vertices $\widehat{C_2}, \widehat{C_3}, \widehat{C_6}$ and $\widehat{C_7}$ is connected to a vertex in a cycle in $C[kp + 2(r + 2)]$. By Lemmas 3.25 and 3.26, these external incident vertices will become all black at time $3r + 1$. Hence, at that time, the situation will be as depicted in Fig. 3. It is easy to verify that within an additional 4 steps all vertices of the cycle will have become black. ∎

Finally,

LEMMA 3.28
The modular construction generates a $(3r+5)$–time irreversible dynamo in $CCC(n)$.

PROOF. Let $\mathcal{C}[j]$ denote the set of all vertices in the cycles in $C[j]$. First observe that, $\forall l_1, l_2 \in \{0, 1, \ldots, 2r+1\}$, $\forall k_1, k_2 \in \{0, 1, \ldots, \frac{n}{p} - 1\}$, the condition $(l_1 \neq l_2)$ or $(k_1 \neq k_2)$ implies $C[kp_1 + 2l_1] \cap C[kp_2 + 2l_2] = \emptyset$; that is, all the cycles considered by the construction are disjoint. Further observe that the union of all the vertices in all the cycles considered by the construction includes all vertices of the $CCC(n)$. Hence, from Lemmas 3.24, 3.25 and 3.26, the theorem follows. ■

We now calculate the size of the dynamos obtained by the modular construction.

LEMMA 3.29
The dynamo generated by the modular construction, has size $n2^{n-4}\frac{4r+5}{r+1}$.

PROOF. *A-Color()* will color black 4 nodes in each cycle belonging to $C[kp]$ $(0 \leq k \leq \frac{n}{p} - 1)$ for a total of $4 \cdot \frac{n}{p} \cdot \frac{2^n}{4}$ black vertices.

B-Color() and *B'-Color()* will color black 2 nodes in each cycle belonging to $C[kp+2l]$ $(0 \leq k \leq \frac{n}{p} - 1$, l: $1 \leq l \leq r$ and $r+2 \leq l \leq 2r+1)$ for a total of $2 \cdot \frac{n}{p} \cdot 2r \cdot \frac{2^n}{4}$ black vertices.

C-Color() color black 1 node in each cycle $C[kp + 2(r+1)]$ $(0 \leq k \leq \frac{n}{p} - 1)$ for a total of $\frac{n}{p} \cdot \frac{2^n}{4}$ black vertices.

Thus, the total number of black vertices is $n2^n\frac{4r+5}{4(4r+4)}$. ■

Summarizing,

THEOREM 3.30
The size of a minimal t-time irreversible dynamo in $CCC(n)$, $t > 5$, is at most $n2^{n-2}(1 + \frac{3}{4t-8})$.

Hence, a $Time \times Size = \Theta(N \cdot t)$ bound follows for minimal t-time irreversible dynamo in $CCC(n)$.

THEOREM 3.31
The size of a minimal t–time irreversible dynamo in $CCC(n)$, $t > 5$, is at least $\lfloor \frac{N}{8} \rfloor$ and at most $\lceil \frac{N}{4}(1 + \frac{3}{4(t-2)}) \rceil$.

PROOF. It follows from Theorem 3.30 and the lower bound of Theorem 3.23. ■

The result of this theorem is somewhat surprising, when compared to the one of Theorem 3.22 for butterfly graphs, since $CCC(n)$ can be embedded into $WBF(n)$ with dilation 2.

The modular construction above produces dynamos with time $t > 5$. For smaller values of t, ad-hoc constructions can be easily found; following are some examples.

THEOREM 3.32
The size of a minimal irreversible stamo in $CCC(n)$ is at least $\lceil \frac{N}{2.5} \rceil$ and at most $\lceil \frac{N}{2} \rceil$.

PROOF. Let $p = 2$ divide n. Consider the following construction: For $0 \leq k \leq \frac{n}{p} - 1$ *A-Color*(kp). Observe that all vertices are in a $C[kp]$ cycle for some k; furthermore, by the property of A-Color, within one time unit all elements in the $C[kp]$ cycles become black. Hence the construction yields a 1–time irreversible dynamo. The size follows from the fact that only half of the nodes are colored black by A-Color. The lower bound is a direct corollary of Theorem 3.6 since $CCC(n)$ is 3-regular. ■

THEOREM 3.33
The size of a minimal 2–time irreversible dynamo in $CCC(n)$ is at most $\lceil \frac{N}{3} \rceil$.

PROOF. Let $p = 3$ divide n. Consider the following construction: For $0 \leq k \leq \frac{n}{p} - 1$ *A-Color*(kp).

First notice that every vertex not in a $C[kp]$ cycle has two neighbours in $C[kp]$, for some k. By the property of A-Color, within one time unit all elements in the $C[kp]$ cycles become black; thus, at time t=2, all vertices are black. In other words, the construction yields a 2–time irreversible dynamo. The size follows from the fact that the $C[kp]$ cycles contain $\frac{2N}{3}$ elements of which only half are colored black. ∎

THEOREM 3.34
The size of a minimal 5–time irreversible dynamo in $CCC(n)$ is at most $\lceil \frac{5N}{16} \rceil$.

PROOF. Let $p = 4$ divide n. Consider the following construction: For $0 \leq k \leq \frac{n}{p} - 1$ *A-Color*(kp) and *C-Color*$(kp + 2)$.

By the property of A-Color, within one time unit all elements of $C[kp]$ become black; hence, at that time, the situations for every cycle in $C[kp + 2]$ is as depicted in Fig. 3. It is easy to verify that within an additional 4 steps all vertices of the cycle will have become black. ∎

3.7 Shuffle–Exchange

Let $\alpha = a_{n-1} \ldots a_1 a_0$ be a binary string. The left cyclic shift and the right cyclic shift operations on α are denoted as $L(\alpha)$ and $R(\alpha)$, respectively, and the shuffle operation $S(\alpha)$ is defined as $S(\alpha) = a_{n-1} \ldots a_1 \hat{a}_0$, where $\hat{a}_0 = 1 - a_0$.

The shuffle exchange graph of dimension n (denoted as $SE(n)$) is a graph (V, E), where $V = \{u \mid u \in \{0,1\}^n\}$ and $E = \{(u, v) \mid R(u) = v \text{ or } L(u) = v \text{ or } S(u) = v\}$.

Due to Theorem 3.2 the lower estimation on the size of a minimal irreversible dynamo in $SE(n)$ is 2^{n-2}. This is close to the following upper bound.

THEOREM 3.35
There exists an irreversible dynamo in $SE(n)$ of size $\lceil \frac{2^n}{3} \rceil$.

PROOF. In the following we will use interchangibly the binary and decimal representations of vertices. Consider a configuration where only the vertices $\lceil 2^n/3 \rceil$,...,$\lceil 2^{n+1}/3 \rceil$ are initially black. Now consider vertices $x < \lceil 2^n/3 \rceil$. We prove that each such a vertex x has two bigger (in decimal representation) neighbours both smaller than $2^{n+1}/3$. We distinguish two cases. Let x be odd, i.e. the binary string is of the form $0\alpha1$. The L-operation leads to $\alpha10$, i.e. $2x$ and the R-operation leads to 10α, i.e. $2^{n-1} + (x - 1)/2$. Let x be even, i.e. of the form $0\alpha0$. Then again the L-operation leads to $2x$ and the S-operation leads to $x + 1$. Following these facts it is clear that it is possible to color vertices $0, ..., \lceil 2^n/3 \rceil - 1$ in decreasing order. The case for remaining vertices is symmetric. ∎

Clearly, the previous dynamo converges in time $\lceil 2^n/3 \rceil$.

LEMMA 3.36
The size of a minimal simple irreversible stamo in $SE(n)$ is at least $\lceil \frac{N}{2.5} \rceil$ and at most $\lceil \frac{N}{2} \rceil$.

PROOF. For the upper bound, decompose the graph into disjoint cycles. In each cycle color every second vertex. The lower bound follows from Theorem 3.6. ∎

THEOREM 3.37
There exists an irreversible dynamo in $SE(n)$ of size $\frac{2^n}{2\sqrt{2}} + o(2^n)$ converging in time n.

PROOF. We color three sets of vertices. The first set is formed by the vertices having equal number of 0's and 1's in their bit-string representations. In the second set there are vertices with less number of 1's of the form $\alpha 1$ and in the third set there are vertices with less number of 0's of the form $\alpha 0$. W.l.o.g. consider the vertices with less number of 1's. Every basic cycle (formed by operation $shift$) contains at least one vertex ending with 1 (and thus colored black). As every white vertex in the cycle has one black neighbour outside the cycle, at least in time n the whole cycle is colored black.

Using $\sum_{k=0}^{n} \binom{2n}{k} \approx \binom{2n}{n} \frac{\sqrt{\pi n}}{2\sqrt{2}}$ and $\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{\pi n}}$ we obtain the result. ∎

3.8 DeBruijn

The *DeBruijn graph of dimension* n (denoted by $DB(n)$) is the graph whose vertices are labeled by all binary strings of length n and whose edges connect each string $a\alpha$, where α is a binary string of length $n-1$ and a is in $\{0,1\}$, with the strings αb, where b is a symbol in $\{0,1\}$.

Question 2: Basic question is to determine the size of a minimal simple irreversible dynamo in DB.

LEMMA 3.38
The size of a minimal simple irreversible stamo in $DB(n)$ is at least $\lceil \frac{N}{3} \rceil$ and at most $\lceil \frac{N}{2} \rceil$.

Due to Theorem 3.4, the size of a minimal t–time simple irreversible dynamo in $DB(n)$ is at least $\lceil \frac{N}{2t+1} \rceil$.

THEOREM 3.39
There is a t–time simple irreversible dynamo in $DB(n)$ of size $\left\lceil \frac{N+1}{4} \cdot \left(1 + \frac{1}{2^t - 1}\right) \right\rceil$.

PROOF. Consider a subgraph induced by vertices $1, ..., N/2-1$. As every vertex x is connected with vertices $2x$, $2x+1$, $\lfloor x/2 \rfloor$, this subgraph forms a binary tree. We have a similar situation for the subgraph induced by $N/2, ..., N-1$ (opposite tree). According to Theorem 3.11 there exists a dynamo of size $\frac{N/2+1}{4}\left(1 + \frac{1}{2^t-1}\right)$ in each of these trees converging in time $t-1$. Consider a configuration given by joining these two dynamos. As every inner vertex from a tree has three neighbours in that tree, all vertices except the leaves can be colored in $t-1$ rounds. As every leaf has at least one neighbour in the opposite tree which is not a leaf, one additional round is sufficient to color all leaves. ∎

THEOREM 3.40
There exists an irreversible dynamo in $DB(n)$ of size 2^{n-2} converging in time at most $2n+1$.

PROOF. Consider the decimal values of the labels. Let dynamo consists of vertices $2^{n-2}, ..., 2^{n-1}$. Every vertex x from $1, ..., 2^{n-2}-1$ is connected with vertices $2x$ and $2x+1$. Thus in time n all vertices smaller than 2^{n-1} are black. Every vertex from $2^{n-1}, ..., 2^{n-1}+2^{n-2}$ has two neighbours smaller than 2^{n-1}. So by the time $n+1$ all vertices smaller than $2^{n-1}+2^{n-2}$ are black. In the next n rounds the rest is colored in a symmetric way. ∎

Question 3: There is an asymptotical gap between the upper and lower bounds known for t–time simple irreversible dynamo in DB and tightening this gap remains an open problem.

Next, we determine the size of irreversible dynamo in DeBruijn with *strong majority rule*.

THEOREM 3.41
The size of a minimal strong irreversible dynamo in $DB(n)$, $n>4$, is at least $\lfloor \frac{2^n}{5} \rfloor$ and at most 2^{n-2}.

PROOF. *Upper bound:* Consider the tree induced by vertices $1, ..., N/2-1$. Color 2^{n-2} vertices from this tree in such a way that every white vertex from the tree has three black vertices from the tree (i.e. every second level). In a single round the tree is recolored black. The remaining vertices form another tree which is then recolored level by level.

Lower bound: comes from Theorem 3.2. ∎

THEOREM 3.42
There exists a t–time strong irreversible dynamo in $DB(n)$ of size $2^{n-2}+\lceil \frac{2^{n+t-1}-1}{2^{t-1}(2^t-1)} \rceil$.

PROOF. Consider the dynamo as in the upper bound of the previous theorem. Color black also every t-th level from the tree induced by $N/2, ..., N$. Single round is needed to recolor the lower tree. And additional $t-1$ rounds are sufficient to recolor the higher tree. In the higher tree we have $x = \lceil \frac{n-1}{t} \rceil$ strips and the number of black vertices is $\sum_{i=0}^{x} 2^{n-t(i+1)}$. ∎

3.9 Hypercubes

An *n-dimensional hypercube* is an undirected graph Q_n consisting of $N=2^n$ vertices and $\frac{1}{2}Nn$ edges. Vertices are labeled by binary strings of length n. There is an edge between two vertices if and only if their labels differ in exactly one bit. Q_n is regular of degree n and has diameter n.

The following result about the lower bound on the size of irreversible dynamo in Q_n is from [11].

THEOREM 3.43
The size of a minimal irreversible dynamo in Q_n is at least $\lceil \frac{2^n}{n+1} \rceil$.

THEOREM 3.44
There exists an irreversible dynamo in Q_n of size $O(2^n/\sqrt{n})$ and time $n/2$.

PROOF. The argument for hypercubes of even dimensions is as follows: take the natural labeling of vertices in n-dimensional hypercube (i.e. by binary strings of length n, where strings of neighbours differ in exactly one bit). Consider the "diagonal" dynamo consisting of vertices having equal number of 0's and 1's in their labels. The adjacent vertex to this dynamo has one bit changed, i.e. it has more black neighbours as white ones. So all neighbours became black in one round. This process can be continued during next $n/2 - 1$ rounds. The size of such dynamo is at most $\binom{n}{n/2} = \frac{n!}{((n/2)!)^2} \approx 2^n/\sqrt{\pi n/2}$.

The bound for hypercubes of odd dimensions comes from considering the "two–diagonals" dynamo consisting of vertices having in their labels the number of 0's greater by one as the number of 1's and vice versa. ∎

The above theorem has been independently observed also by S. Dobrev and F. Geurts.

Question 4: There is still an asymptotical difference between the upper and lower bounds on the size of minimal dynamos in hypercubes. The open problem is to tighten this difference.

THEOREM 3.45
The size of a minimal t–time irreversible dynamo in the n-dimensional hypercube Q_n is at least $\lceil \frac{N}{2t+1} \rceil$ and at most $\lceil \frac{N}{t+1} \rceil - \lceil \frac{N}{\sqrt{\pi n/8}} \rceil$ for $t = o(\sqrt{n})$.

PROOF. The upper bound follows from $\sum_{i=1}^{t} \binom{n}{k+i} > t.\binom{n}{k}$ for $k \geq 1$ and $k + t < n/2$. ∎

COROLLARY 3.46
The size of a minimal irreversible stamo in Q_n is at least $\lceil N/3 \rceil$ and at most $N/2 - o(N)$.

4 Irreversible Model versus Reversible Monotone Model

Recall that by the reversible monotone dynamo we mean a set of vertices M such that:

- in round 0 vertices from M are black and other vertices are white;
- in round $t > 0$ a vertex v is black if and only if it had at least $\lceil deg(v)/2 \rceil$ black neighbours in round $t - 1$;
- a black vertex never becomes white;

- in some round t_m all vertices are black.

A subgraph H, $H \subset G$, is called a *black (or simple) immune subgraph* if every vertex $v \in H$ has at least $\lceil deg(v)/2 \rceil$ neighbours in H.

LEMMA 4.1
A set of vertices M is a (reversible) monotone dynamo if and only if M is irreversible dynamo and is black immune.

On some regular topologies we get asymptotically tight bounds on the size of minimal monotone dynamos:

LEMMA 4.2
For every irreversible dynamo M in an n-dimensional wrapped butterfly, CCC, DeBruijn there exists a reversible monotone dynamo M', $M' \supseteq M$, with at most $4|M|$, $8|M|$, $4|M|$ vertices, respectively.

PROOF. In butterfly and CCC, the cycles of length 4 and 8, respectively, are black immune. In DeBruijn, consider cycles $axb \to xb\overline{a} \to \overline{a}xb \to xba \to axb$ for $0...0 \neq xb\overline{a} \neq 1...1$ (if $axb = xba$ the cycle has length 3) and cycles $a\overline{a}...\overline{a} \to aa\overline{a}...\overline{a} \to a\overline{a}...\overline{a}a \to \overline{a}a\overline{a}...\overline{a} \to a\overline{a}...\overline{a}$ otherwise. ∎

LEMMA 4.3
The size of a minimal monotone dynamo on a binary complete tree $T_{2,h}$ is at most $5 \cdot 2^{h-2}$.

LEMMA 4.4
There exists a monotone dynamo on Q_n of size $O(2^n/\sqrt{n})$.

Question 5: The question is whether in hypercubes the size of a minimal monotone t–time dynamo is asymptotically the same as the size of minimal irreversible t–time dynamo.

As indicated in [4], the size of a minimal irreversible dynamo in n-dimensional tori is smaller by a factor two than the size of a minimal reversible monotne dynamo. This raised the intriguing question of whether it is always possible to transform an irreversible dynamo into a monotone one, increasing the number of initially black vertices by at most constant times. We answer this hypothesis in a negative way:

THEOREM 4.5
There is a planar graph $G = (V, E)$ with $|V| = N$ such that the size of a minimal simple irreversible dynamo on G is 1, but the size of any minimal simple reversible monotone dynamo on G is at least $\lfloor \frac{N}{3} \rfloor$.

PROOF. Consider the following graph $G = (V, E)$, $V = \{0, ..., N\}$, $E = \{(N, 0)\} \cup \{(i, i+1) \mid 0 \leq i \leq N-1\} \cup \{(0, 3i) \mid 1 \leq i \leq \lfloor \frac{N}{3} \rfloor\}$. Irreversible dynamo is $\{0\}$. Reversible monotone dynamo is $\{0, 1\} \cup \{3i, 3i+1 \mid i = 1, ..., \lfloor \frac{N}{6} \rfloor\}$. Clearly, each reversible monotone dynamo on G is of size at least $\lfloor \frac{N}{3} \rfloor$. ∎

The question remains open for regular graphs.

Question 6: The question is whether the ratio between the size of reversible monotone and irreversible dynamo is $\Omega(N)$ also for regular graphs.

Note that the above hypothesis remains false even if $Size$ is replaced by $Size \times Time$. Take an N-vertex star graph (i.e. a center vertex with $N-1$ neighbours of degree 1); the expression $Size \times Time$ is 1 for irreversible model and $N-1$ for reversible monotone model.

5 Graphs having Slow Complete Computations from Minimal Dynamos

Another interesting topic is to search for graphs in which each complete computation starting from a minimal irreversible dynamo is in certain sense slow.

We are aware of a graph having the property of "slow computation" from an arbitrary minimal irreversible dynamo (i.e. a complete computation which is asymptotically slower than the diameter of the graph). In Torus Serpentinus [4] of size $n \times m$ (with n - the length of serpentines being asymptotically lower than m - the number of serpentines), each minimal dynamo of size n converges in time at least $n \cdot m$, while the dynamo of size $n+m$ converges in time n. The diameter of a $n \times m$ Torus Serpentinus is $n+m$.

Another interesting example is the butterfly $BF(n)$. There are minimal dynamos (of size 2^{n-2}) with complete computations of time $diameter(G)$ (see [7]), but there also exists a dynamo of the same size 2^{n-2} with a complete computation of time $\frac{3}{2}diameter(G) - 2$. This dynamo consists of nodes in column $n-2$ (note that the columns are numbered $0, ..., n$ from right to left) and rows $\{0, 2, ..., 2^{n-2} - 2\}$ and $\{2^{n-2}+1, 2^{n-2}+3, ..., 2^{n-1}-1\}$. For the computation to proceed to the column $n-1$, both subbutterflies in columns $0, ..., n-2$ and rows $0, ..., 2^{n-2}-1$ and $2^{n-2}, ..., 2^{n-1}-1$ must be colored. This is done in $2(n-2)$ steps (as in the previous dynamo). Additional $2+n$ steps are needed to color the rest of the graph. Thus the dynamo works in $3n-2 = \frac{3}{2}diameter(G) - 2$ steps. The time of a complete computation depends in this case on the structure of dynamos.

Question 7: The main open problem is whether for some specific topologies (e.g. hypercubes) $\omega(diameter(G))$-time irreversible dynamos are asymptoticaly smaller than $O(diameter(G))$-time irreversible dynamos.

In other words the question is whether in these topologies there are "small" dynamos with "slow" complete computations or there always exists a "minimal" and "fast" dynamo.

6 Conclusions

We have initiated the study of dynamos with respect to both the time and the size. We derived tight time–size tradeoffs for rings, tori, wrapped butterflies, cube connected cycles and hypercubes, and asymptotically optimal size bounds for simple (strong) irreversible dynamos in butterflies and shuffle–exchange (DeBruijn).

In future research we are concentrating on the following topics:

- Unsolved questions
  - Determine the size of minimal irreversible dynamos for wrapped butterflies, De-Bruijn and hypercubes.
  - Determine the size of minimal t-time irreversible dynamos for DeBruijn.
- Upper bounds related to subsection 3.1 (for more general classes of graphs).
- Characterizations of graphs for which the time of a complete computation from a minimal irreversible dynamo is bounded by the diameter of the graph.

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